Two hypergeometric summation formulae related to 9-j coefficients

1994 J. Phys. A: Math. Gen. 27 6943

(http://iopscience.iop.org/0305-4470/27/20/029)

J. Phys. A: Math. Gen. 27 (1994) 6943–6945. Printed in the UK

COMMENT

Two hypergeometric summation formulae related to 9-j coefficients

Per W Karlsson

Mathematical Institute, The Technical University of Denmark, DK-2800 Lyngby, Denmark

Received 12 July 1994, in final form 14 August 1994

Abstract. Two summation formulae for special Kampé de Fériet functions $F^{0:3}_{1:1}$ with variables $(1, 1)$ are proved.

In a recent article in this journal [3], Van der Jeugt *et al*, while studying 9-j coefficients, obtained several summation formulae for various Kampé de Fériet hypergeometric functions. Continuing [3], we shall give further consideration to two of these results by proving one of them and generalizing the other. Both formulae involve the Kampé de Fériet function defined by

$$F^{0:3}_{1:1}\left[\begin{matrix} :a,b,c\,;a',b',c' \\ d:\quad e\quad ;\quad e' \end{matrix}\,\middle|\, x,y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_m(c)_m(a')_n(b')_n(c')_n}{(d)_{m+n}(e)_m(e')_n} \frac{x^m y^n}{m!n!}. \tag{1}$$

The region of convergence is given by $|x| < 1$ and $|y| < 1$; and the series is (cf [1]) absolutely convergent for $|x| = 1 = |y|$, provided that we have

$$\mathrm{Re}(d + e - a - b - c) > 0 \quad \wedge \quad \mathrm{Re}(d + e' - a' - b' - c') > 0.$$

Note the natural modifications if there are negative integral values of one or several numerator parameters. It is, as usual, understood that no denominator parameter is zero or a negative integer.

The two summation formulae read as follows,

$$F^{0:3}_{1:1}\left[\begin{matrix} :a,b,c\,;a+c+e'-e,b+c+e'-e,-c \\ a+b+c+e'-e:\quad e\quad ;\quad e' \end{matrix}\,\middle|\, 1,1\right]$$
$$= \Gamma\left[\begin{matrix} e,e' \\ e-c,e'+c \end{matrix}\right] \qquad (\mathrm{Re}(e) > 0,\ \mathrm{Re}(e') > 0) \tag{2}$$

and

$$F^{0:3}_{1:1}\left[\begin{matrix} :\ a,-m-p,e+m\ ;\ d-a,d+m+p,c' \\ d\ :\quad e\quad ;\quad d-a+p+1 \end{matrix}\,\middle|\, 1,1\right]$$
$$= \Gamma\left[\begin{matrix} d-a+1,1-c' \\ d-a+1-c' \end{matrix}\right] \frac{(d-a)_p(d-a+1)_p(a)_m(-1)^m(m+p)!}{(d-a+1-c')_p(d)_{m+p}(e)_m p!}$$
$$(m,p \in \mathbb{N}_0,\ \mathrm{Re}(1-m-c') > 0). \tag{3}$$

0305-4470/94/206943+03$19.50

For convenience, we use Slater's notation for a quotient of products of Gamma functions [2, section 2.1.1].

To establish these formulae, we consider the Eulerian integral representation

$$\Gamma\left[\begin{matrix} c, e-c, c', e'-c' \\ e, e' \end{matrix}\right] F^{0:3}_{1:1}\left[\begin{matrix} :a,b,c\,;a',b',c' \\ d:\quad e\quad;\quad e' \end{matrix}\middle| x,y\right]$$
$$= \int_0^1\int_0^1 s^{c-1}(1-s)^{e-c-1}t^{c'-1}(1-t)^{e'-c'-1} F_3[a,a';b,b';d;xs,yt]\,\mathrm{d}s\,\mathrm{d}t$$
$$(\mathrm{Re}(e) > \mathrm{Re}(c) > 0,\ \ \mathrm{Re}(e') > \mathrm{Re}(c') > 0) \qquad (4)$$

and proceed in a number of steps. First, set $x = 1 = y$, $a' = d - a$, $b' = d - b$, and apply the well known reduction formula

$$F_3[a, d-a; b, d-b; d; s, t] = (1-t)^{a+b-d}\,{}_2F_1[a, b; d; s+t-st].$$

Next, introduce new variables $\sigma = 1 - s$, $\tau = 1 - t$, such that the variable of the ${}_2F_1$ is $1 - \sigma\tau$, and transform to variable $\sigma\tau$ by a classical analytic continuation formula. In this way we arrive at a sum of two terms, each involving an Eulerian integral for ${}_4F_3[1]$. In fact, we find the intermediate result,

$$F^{0:3}_{1:1}\left[\begin{matrix} :a,b,c\,;d-a,d-b,c' \\ d:\quad e\quad;\quad e' \end{matrix}\middle| 1,1\right]$$
$$= \Gamma\left[\begin{matrix} e', d-a-b, d, e'+a+b-c'-d \\ e'-c', d-a, d-b, e'+a+b-d \end{matrix}\right]$$
$$\times\ {}_4F_3\left[\begin{matrix} a, b, e-c, e'+a+b-c'-d \\ a+b-d+1, e, e'+a+b-d \end{matrix}\middle| 1\right]$$
$$+ \Gamma\left[\begin{matrix} e, d, a+b-d, e+d-a-b-c \\ e-c, a, b, e+d-a-b \end{matrix}\right]$$
$$\times\ {}_4F_3\left[\begin{matrix} d-a, d-b, e'-c', e+d-a-b-c \\ d-a-b+1, e', e+d-a-b \end{matrix}\middle| 1\right]$$
$$(\mathrm{Re}(d+e-a-b-c) > 0,\ \mathrm{Re}(e'-d+a+b-c') > 0,\ \mathrm{Re}(1-d-c-c') > 0). \qquad (5)$$

In (5), we now set $c' = -c$, and $d = a + b + c + e' - e$. Then, the ${}_4F_3[1]$'s become ${}_2F_1[1]$'s, to which Gauss' summation theorem is applied. Finally, after a few further steps, including the use of elementary properties of the Gamma function, we arrive at the reduction formula (2), which was conjectured (and numerically checked) in [3], but proved only for terminating series.

To establish the reduction formula (3), we consider (5) together with a different set of conditions:

$$-b \in \mathbb{N}_0 \quad\wedge\quad e' = d-a+1-b+e-c \quad\wedge\quad -(e-c) \in \mathbb{N}_0.$$

When we insert the first and the second of these conditions, the right-hand member of equation (5) becomes,

$$\Gamma\left[\begin{matrix} d-a-b-c+e+1, d-a-b, d, e-c-c'+1 \\ d-a-b-c-c'+e+1, d-a, d-b, e-c+1 \end{matrix}\right]$$
$$\times\ {}_4F_3\left[\begin{matrix} a, b, e-c, e-c-c'+1 \\ a+b-d+1, e, e-c+1 \end{matrix}\middle| 1\right].$$

This ${}_4F_3[1]$ is a polynomial, and we may obviously let $e - c$ tend to a non-positive integer $-m$ in each term, taking the factor $(\Gamma(e - c + 1))^{-1}$ into account. As a result, only the mth term survives; and after a few more manipulations it will be seen that the expression is in fact zero for $m > -b$. We then set b equal to $-(m + p)$ and arrive at the reduction formula (3) after a few further steps. The particular case $c' = -(m + r)$, where r is an integer, was given in [3].

The author is indebted to J Van der Jeugt for fruitful discussions in conjunction with [3] before its publication.

References

[1] H'ai N T, Marichev O I and Srivastava H M 1992 A note on the convergence of certain families of multiple hypergeometric series *J. Math. Anal. Appl.* **164** 104–15

[2] Slater L J 1966 *Generalized Hypergeometric Functions* (Cambridge: Cambridge University Press)

[3] Van der Jeugt J, Pitre S N and Srinivasa Rao K 1994 Multiple hypergeometric functions and 9-*j* coefficients *J. Phys. A: Math. Gen.* **27** 5251–64